Taylor's Inequality (error bound):
On a given interval [a,b],
if $|f^{(n+1)}(x)| \leq M$, then

$$|f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x-b|^{n+1}$$

Recall:

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(b)(x-b)^k$$

$$= \frac{1}{0!}\underbrace{f(b)}_{0} + \frac{1}{1!}\underbrace{f'(b)}_{1}(x-b) + \frac{1}{2!}\underbrace{f''(b)}_{0}(x-b)^2 + \cdots + \frac{1}{n!}f^{(n)}(b)(x-b)^n$$

Entry Task:
Find the 7th Taylor polynomial for $f(x) = \sin(x)$, based at $b = 0$.
Find a bound on the error over the interval [-3,3].

$$f(x) = \sin(x) \qquad f(0) = 0$$

$$f'(x) = \cos(x) \qquad f'(0) = 1$$

$$f''(x) = -\sin(x) \qquad f''(0) = 0$$

$$f'''(x) = -\cos(x) \qquad f'''(0) = -1$$

$$f^{(4)}(x) = \sin(x) \qquad f^{(4)}(0) = 0$$

REPEAT

STOP!

$$\sin(x) \approx 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + 0 - \frac{1}{7!}x^7$$

$$\boxed{T_7(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7}$$

ERROR BOUND: $|f^{(8)}(x)| = |\sin(x)| \leq 1$

$$\text{ERROR} \leq \frac{1}{8!}|x|^8 \leq \frac{1}{8!}3^8 \approx 0.1627$$

$$\sin(x) \approx T_7(x) \quad \pm 0.1627$$

ON [-3, 3]

TN 4: Taylor Series

Def'n:

The **Taylor Series** for f(x) based at b is

$$\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(b)(x-b)^k = \lim_{n\to\infty} T_n(x)$$

If the limit exists at a particular *x*, then we say the series **converges** at *x*. Otherwise, we say it **diverges** at x.

The **open interval of convergence** is the largest open interval of values over which the series converges.

Note:

If

$$\lim_{n\to\infty} \frac{M}{(n+1)!} |x-b|^{n+1} = 0$$

then the error goes to zero and *x* is in the open interval of convergence.

NOTE for $\sin(x)$ on $[-3,3]$ the error bound would look like

$$\frac{1}{(n+1)!} 3^{n+1} \to 0 \quad \text{as } n \to \infty$$

Ex)

$$\frac{3^{100}}{100!} = \frac{3 \cdot 3 \cdots 3 \cdot 3}{100 \cdot 99 \cdots 3 \cdot 2} \leftarrow \text{much, much larger}$$

A few patterns we now know:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \quad \Rightarrow \quad e^x = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \quad \Rightarrow \quad \sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \quad \Rightarrow \quad \cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}$$

These three converge for ALL values of *x*. So the **open interval of convergence** for each series above is (-∞,∞)

NOTE :

	$k=0$	$k=1$	$k=2$	$k=3$		
$(-1)^k$	$+1$	-1	$+1$	-1	...	ALTERNATING SIGN
$2k+1$	1	3	5	7	...	ODDS
$2k$	0	2	4	6	...	EVENS

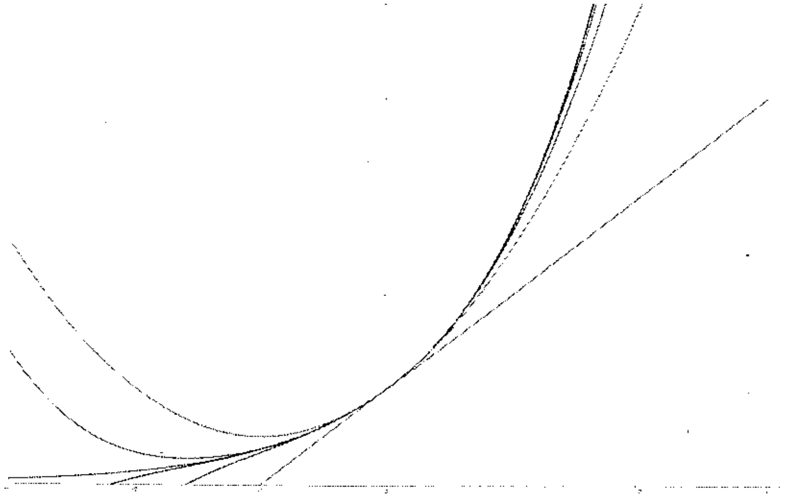
AWESOME!

ASIDE $\boxed{e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots}$

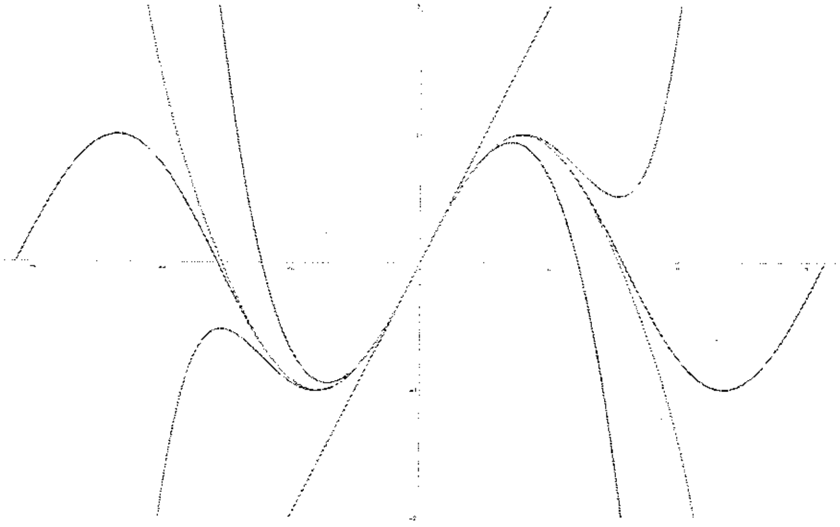
$$\boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{\pi}{4}\right)^{2k+1} = \frac{\sqrt{2}}{2}}$$ COOL!

Visuals of Taylor Polynomials:

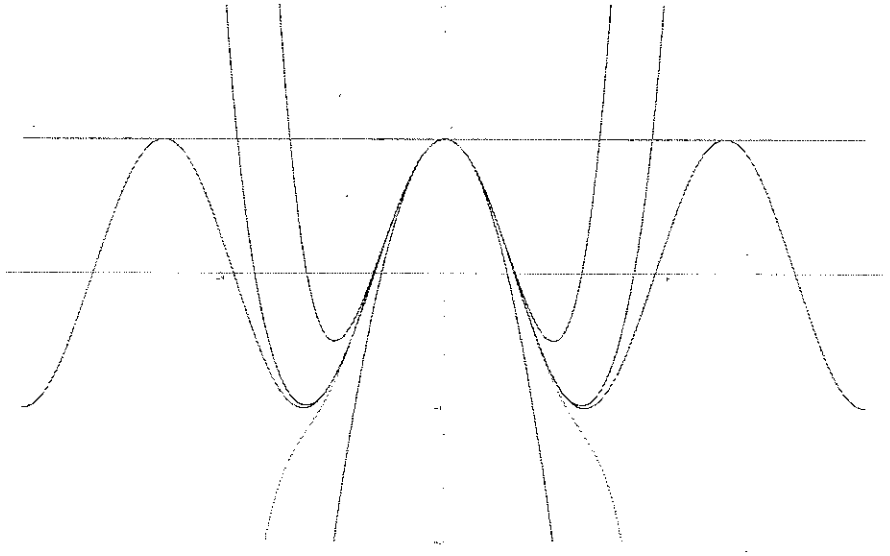
1. $f(x) = e^x$ as well as $T_1(x)$, $T_2(x)$, $T_3(x)$, $T_4(x)$ and $T_5(x)$ are shown:



2. $f(x) = \sin(x)$ as well as $T_1(x)$, $T_3(x)$, $T_5(x)$, and $T_7(x)$ are shown:



3. $f(x) = \cos(x)$ as well as $T_1(x)$, $T_2(x)$, $T_4(x)$, $T_6(x)$, and $T_8(x)$ are shown:



Now consider $f(x) = \frac{1}{1-x}$ based at 0.
Find the 10th Taylor polynomial.
What is the error bound on [-1/2,1/2]?
What is the error bound on [-2,2]?

$$f(x) = (1-x)^{-1} \Rightarrow f(0) = 1$$

$$f'(x) = -(1-x)^{-2} \Rightarrow f'(0) = 1$$

$$f''(x) = 2(1-x)^{-3} \Rightarrow f''(0) = 2$$

$$f'''(x) = 2 \cdot 3(1-x)^{-4} \Rightarrow f'''(0) = 3!$$

$$f^{(4)}(x) = 2 \cdot 3 \cdot 4\,(1-x)^{-5} \Rightarrow f^{(4)}(0) = 4!$$

etc. . .

$$T_{10}(x) = 1 + \frac{1}{1!}1\,x^1 + \frac{1}{2!}2\,x^2 + \frac{1}{3!}3!\,x^3 + \frac{1}{4!}4!\,x^4 + \cdots + \frac{1}{10!}10!\,x^{10}$$

$$= 1 + x + x^2 + x^3 + \cdots + x^{10}$$

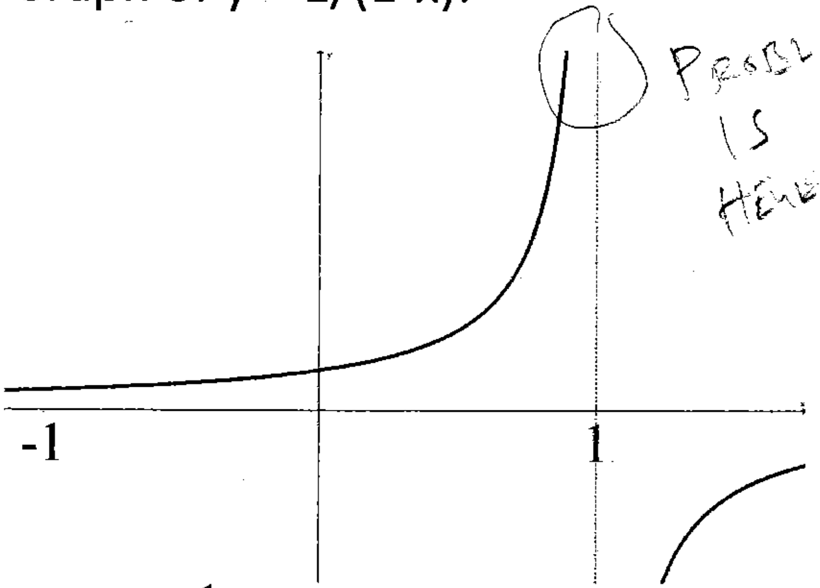
$$f^{(11)}(x) = 11!\,(1-x)$$

$$= \frac{11!}{(1-x)^{12}} \le M \quad ???$$

$$\left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \Rightarrow M = \frac{11!}{(1/2)^{12}}$$

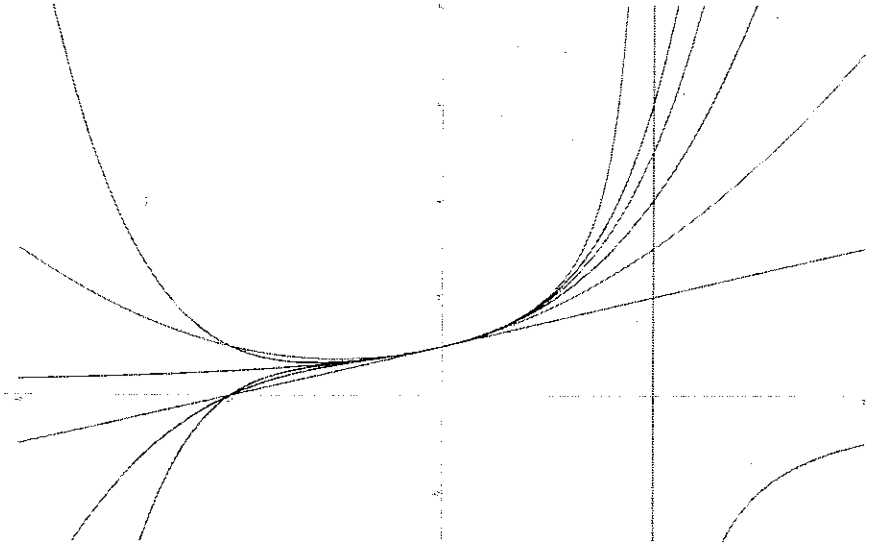
[-2, 2] ⇒ NO BOUND!!!
INFINITY
(ASYMPTOTE AT $x = 1$)

Graph of y = 1/(1-x):

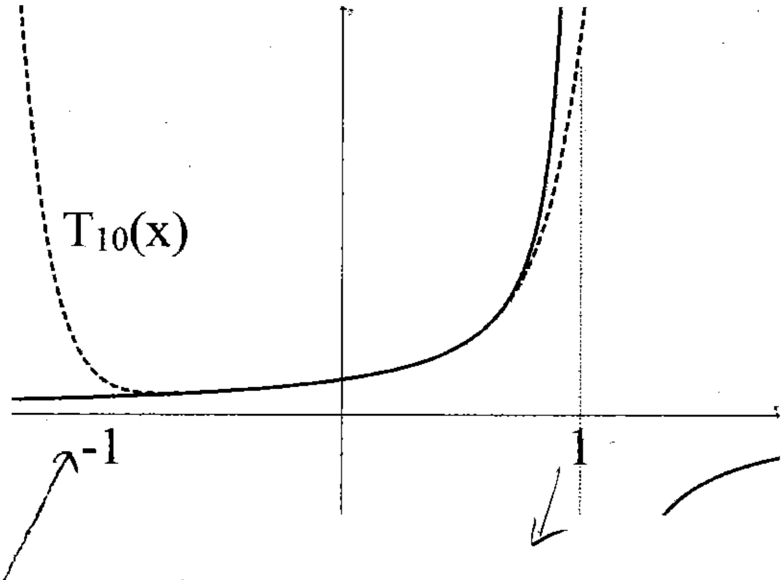


PROBLEM IS HERE

$f(x) = \frac{1}{1-x}$ as well as $T_1(x)$, $T_2(x)$, $T_3(x)$, $T_4(x)$, and $T_5(x)$ are shown:



Graph of $f(x) = \frac{1}{1-x}$ and $T_{10}(x)$:



EVEN THOUGH THERE IS NO ASYMPTOTE HERE

SYMMETRY CAUSES THE TAYLOR POLYNOMIAL BASED AT b=0 TO BE "STEEP" HERE AS WELL (ONE AWAY ON BOTH SIDES)

By Friday, we discuss all the following:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \quad \Rightarrow \quad \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$\left.\begin{aligned} -\ln(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots \quad &\Rightarrow \quad -\ln(1-x) = \sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1} \\ \arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \cdots \quad &\Rightarrow \quad \arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \end{aligned}\right\} \text{Next time}$$

The open interval of convergence for all three of these series: **-1 < x < 1**.

Sigma Notation Notes

Definition:

$$\sum_{k=a}^{b} f(k) = f(a) + f(a+1) + f(a+2) + \cdots + f(b-1) + f(b)$$

You try: Expand these

$$\sum_{i=1}^{3} \frac{(-1)^i}{i^2} x^i = \frac{(-1)^1}{1^2} x^1 + \frac{(-1)^2}{2^2} x^2 + \frac{(-1)^3}{3^2} x^3 = -x + \frac{1}{4}x^2 - \frac{1}{9}x^3$$

↕ SAME!!!! DIFFERENT WAY TO SUMMARIZE THE SAME PATTERN!

$$\sum_{k=13}^{15} \frac{(-1)^{(k-12)}}{(k-12)^2} x^{k-12} = \frac{(-1)^1}{1^2} x^1 + \frac{(-1)^2}{\underset{\overset{\uparrow}{(14-12)}}{2^2}} x^2 + \frac{(-1)^3}{\underset{\overset{\uparrow}{(15-12)}}{3^2}} x^3 = -x + \frac{1}{4}x^2 - \frac{1}{9}x^3$$

Note: In the examples, above *i* and *k* are *dummy* variables, used to summarize a pattern.

Constants and adding:

Expand then combine

$$5\sum_{k=2}^{4} k^2x^k - 6\sum_{k=2}^{4}\frac{1}{k!}x^k$$

$$= 5\left((2)^2x^2 + (3)^2x^3 + (4)^2x^4\right) - 6\left(\frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4\right)$$

$$= 5\cdot(2)^2x^2 + 5(3)^2x^3 + 5(4)^2x^4 - \frac{6}{2!}x^2 - \frac{6}{3!}x^3 - \frac{6}{4!}x^4$$

$$= \left(5\cdot(2)^2 - \frac{6}{2!}\right)x^2 + \left(5(3)^2 - \frac{6}{3!}\right)x^3 + \left(5(4)^2 - \frac{6}{4!}\right)x^4$$

SAME AS

$$\sum_{k=2}^{4}\left(5k^2x^k - \frac{6}{k!}x^k\right) = \sum_{k=2}^{4}\underbrace{\left(5k^2 - \frac{6}{k!}\right)}_{a_k}x^k$$

Summary: For adding/subtracting and constant multiples, you can manipulate in the same way you learned to manipulate integrals.

Derivatives and Integrals

Recall:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C\,, \qquad \frac{d}{dx}(x^n) = nx^{n-1}$$

Thus,

To differentiate a Taylor series → change x^k to kx^{k-1}

To integrate a Taylor series → change x^k to $\frac{1}{k+1}\ x^{k+1}$

Example: Find the derivative and general antiderivative of

$$f(x) = -x + \frac{1}{8}x^2 - \frac{1}{27}x^3 + \frac{1}{64}x^4 - \frac{1}{125}x^5 \qquad = \sum_{k=1}^{5} \frac{(-1)^k}{k^3} x^k$$

$$f'(x) = -1 + \frac{1}{4}x - \frac{1}{9}x^2 + \frac{1}{16}x^3 - \frac{1}{25}x^4 \qquad = \sum_{k=1}^{5} \frac{(-1)^k}{k^3}\,k\,x^{k-1}$$

$$f'(x) = \boxed{\sum_{k=1}^{5} \frac{(-1)^k}{k^2}\,x^{k-1}}$$

$$\int f(x)dx = C - \frac{1}{2}x^2 + \frac{1}{3}\frac{1}{8}x^3 - \frac{1}{4}\frac{1}{27}x^4 + \frac{1}{5}\cdot\frac{1}{64}x^5 - \frac{1}{6}\frac{1}{125}x^6$$

$$\boxed{\int f(x)dx = C + \sum_{k=1}^{5} \frac{(-1)^k}{k^3}\frac{1}{(k+1)}x^{k+1}}$$